Local Solubility of quadratic forms over biprojective base

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The Set-up

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Suppose $V \subseteq \mathbb{P}^m \times \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^n$ are smooth projective varieties over $\mathbb{Q}$ and that $\pi : V \to Y$ is a dominant map, whose generic fibre is geometrically integral. Writing $\mathbf{a} \in \mathbb{P}^n(\mathbb{Q})$ as $\mathbf{a} = [a_0 : \ldots : a_n]$ such that $a_0, \ldots, a_n \in \mathbb{Z}$, $\gcd(a_0, \ldots, a_n) = 1$, we set $\|\mathbf{a}\| = \max\{|a_0|, \ldots, |a_n|\}$.

We are interested in the quantities

$$N_{glob}(B; Y) = \sharp \left\{ \mathbf{a} \in Y(\mathbb{Q}) : \pi^{-1}(\mathbf{a})(\mathbb{Q}) \neq \emptyset, \ \|\mathbf{a}\| \leq B \right\};$$

$$N_{loc}(B; Y) = \sharp \left\{ \mathbf{a} \in Y(\mathbb{Q}) : \pi^{-1}(\mathbf{a}) \text{ is ELS}, \ \|\mathbf{a}\| \leq B \right\}.$$

[ELS = everywhere locally soluble]

Examples

Let $V : a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz = 0$, $Y = \mathbb{P}^5$ and define $\pi : V \to Y$ as $\pi(\mathbf{x}, \mathbf{a}) = \mathbf{a}$. Thus, the fibres of π form the family of ternary quadratic forms.

Theorem (J. P. Serre, 1990)

$$N_{glob}(B; Y) \ll \frac{B^6}{\sqrt{\log B}}.$$

The HP holds in this case, and so one may equivalently study $N_{loc}(B; Y)$. Serre conjectured matching lower bounds. This was proven in 2007 by Hooley.

Parameterisation by Projective Space

More generally, for $Y = \mathbb{P}^n$, we have the following:

Theorem (Loughran–Smeets, (2016))

For $(V, \mathbb{P}^n, \pi)$ *as before,*

$$N_{loc}(B; \mathbb{P}^n) \ll \frac{B^{n+1}}{(\log B)^{\Delta(\pi)}}$$

where $\Delta(\pi)$ *is an invariant defined by the Galois action on the irreducible components of the fibres of* π*.*

This bound is conjecturally sharp in cases where every fibre of π has an irreducible component of multiplicity 1.

Parameterisation by Projective Space

For $(V, \mathbb{P}^n, \pi)$

- (Poonen–Voloch (2004), Loughran–Smeets (2016)) When $\Delta(\pi) = 0$, a positive density of fibres, $\pi^{-1}(\mathbf{a})$, are ELS.
- (Loughran–Matthiesen (2019)) Lower bounds of order $B^2/(\log B)^{\Delta(\pi)}$ for general V when $Y = \mathbb{P}^1$, and for conic bundles and families of multi-norm equations over $\mathbb{P}^n$.
- (Loughran–Rome–Sofos (2022)) Asymptotics when $(V, \mathbb{P}^2, \pi)$ represents the family of diagonal plane conics:

$$N_{glob}(B, \mathbb{P}^2) = N_{loc}(B, \mathbb{P}^2) \sim \frac{cB^3}{(\log B)^{3/2}}$$

for some $c > 0$.

Parameterisation by a Quadric (1)

What about more general Y? Particularly Y a Fano variety satisfying certain non-accumulation assumptions and such that $Y(\mathbb{Q})$ is Zariski dense in Y. The definition of $\Delta(\pi)$ also generalises for such bases. We may therefore conjecture that for a triple (V, Y, π) such that all fibres of π have an irreducible component of multiplicity 1,

$$N_{loc}(B; Y) \sim \frac{c_Y \# \{\mathbf{y} \in Y(\mathbb{Q}) : \|\mathbf{y}\| \leq B\}}{(\log B)^{\Delta(\pi)}},$$

for some constant $c_Y > 0$.

Parameterisation by a Quadric (2)

- (Loughran (2018),Loughran–Takloo-Bighash–Tanimoto (2020)) Asymptotics when Y is an algebraic group or a wonderful compactification.
- (Browning–Loughran (2018)) established expected upper bounds when families are parameterised by a smooth quadric of dimension ≥ 3.
- (Sofos–Visse-Martindale (2019)) Conic fibres over a hypersurface.
- (Browning–Heath-Brown (2020)) established expected densities when $\Delta(\pi) = 0$ for quadratic forms with rank ≥ 5.

A Quadric of Dimension 2

In 2023 Browning–Lyczak–Sarapin studied the case where $V \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ is defined by the equations

$$V : y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 = 0, \ y_0 y_1 = y_2 y_3,$$

and $Y : y_0 y_1 = y_2 y_3 \subseteq \mathbb{P}^3$. Here,

$$\sharp \{ \mathbf{y} \in Y(\mathbb{Q}) : \|\mathbf{y}\| \leq B \} \sim cB^2 \log B$$

and it may be shown that $\Delta(\pi) = 2$. Our expectation then gives

$$N_{glob}(B; Y) = N_{loc}(B; Y) \sim \frac{c' B^2}{\log B}.$$

Disproving the Naive Expectation

Contrary to this expectation we have:

Theorem (Browning–Lyczak–Sarapin, 2023)

$$B^2 \ll \sharp \left\{ \mathbf{y} \in \mathbb{Z}^4 : \begin{array}{c} y_0 y_1 = y_2 y_3, \ \gcd(y_0, y_1, y_2, y_3) = 1, \\ \sum_{i=0}^{3} y_i x_i^2 = 0 \textit{ has } \mathbb{Q} \textit{ point}, \ \|\mathbf{y}\| \leq B \end{array} \right\} \ll B^2$$

The lower bound is obtained by noting that the points on Y where

$$-y_0 y_2 = n^2$$

for some $n \in \mathbb{Z}$, always give a fibre with a rational point. What happens if we remove these points?

Main Theorem

Theorem (W., 2024)

$$\sharp \left\{ \mathbf{y} \in \mathbb{Z}^4 : \begin{array}{c} y_0 y_1 = y_2 y_3, \\ -y_0 y_2 \neq \square, -y_0 y_3 \neq \square, \\ \gcd(y_0, y_1, y_2, y_3) = 1, \\ \sum_{i=0}^{3} y_i x_i^2 = 0 \textit{ has } \mathbb{Q} \textit{ point}, \\ \|\mathbf{y}\| \leq B \end{array} \right\} \sim \frac{c' B^2 \log\log B}{\log B}$$

for some constant $c' > 0$.

$\log\log B$??????

Suppose X is a smooth projective variety, H some height function. **Batyrev–Manin–Tschinkel Conjecture**: There exists a thin set T such that

$$\#\{x \in (X \setminus T)(\mathbb{Q}) : H(x) \le B\} \sim cB^a(\log B)^{b-1}$$

When H is the anticanonical height, b is the Picard rank of X. **Loughran–Smeets Conjecture** (for projective space): Suppose we have a triple $(V, \mathbb{P}^n, \pi)$ as before. Then

$$\#\{\mathbf{a} \in \mathbb{P}^n(\mathbb{Q}) : \|\underline{a}\| \le B,\ \pi^{-1}(\mathbf{a})(\mathbb{Q}) \neq \emptyset\} \sim \frac{cB^{n+1}}{(\log B)^{\Delta(\pi)}}$$

We also have conjectures for more general bases, though these also only include B^a and $(\log B)^b$. This conjecture does not cover $Y : y_0y_1 = y_2y_3$.

Method (1)

So where does the $\log\log B$ come from? Analytically, it occurs as a consequence of the hyperbola method. Using the parameterisation of $Y : y_0 y_1 = y_2 y_3$ by $\mathbb{P}^1 \times \mathbb{P}^1$, the counting problem becomes

$$\frac{1}{4}N(B) = \frac{1}{4}\sharp\left\{ \mathbf{t} \in (\mathbb{Z}/\{0\})^4 : \begin{array}{c} X(\mathbf{t}) \text{ has a } \mathbb{Q} \text{ point}, \\ \gcd(t_0, t_1) = \gcd(t_2, t_3) = 1 \\ -t_0 t_1, -t_2 t_3 \neq \square \\ \color{red}{\|t_0, t_1\| \cdot \|t_2, t_3\|} \leq B \end{array} \right\}$$

where $X(\mathbf{t})$ is the diagonal quadric surface given by

$$t_0 t_2 x_0^2 + t_1 t_3 x_1^2 + t_1 t_2 x_2^2 + t_0 t_3 x_3^2 = 0.$$

Method (2)

Using Jacobi symbols to detect local solubility of each $X(\mathbf{t})$, we can write this counting problem as a character sum with the given hyperbolic height. The main term of this sum is given by the following:

Lemma (W.)

Let $B \geq 3$. *Then*

$$\sum_{\substack{\mathbf{l}\in\mathbb{N}^4 \\ \|l_0,l_1\|\cdot\|l_2,l_3\|\leq B}} \frac{\mu^2(2l_0)\mu^2(2l_1)\mu^2(2l_2)\mu^2(2l_3)}{\tau(l_0)\tau(l_1)\tau(l_2)\tau(l_3)} \sim \frac{c' B^2 \log\log B}{\log B}.$$

for some constant $c' > 0$.

This is proven using a Tauberian theorem and the hyperbola method. Note that $\tau(n) = \#\{d|n\}$.

Variations of the Large Sieve

The error terms of the character sums are bounded using classical Tauberian theorems and variations of the large sieve for quadratic characters.

Theorem (Heath-Brown (1995))

Suppose a_n and b_m are any sequences in $\mathbb{C}$ supported on odd square-free integers such that $|a_n|, |b_n| \leq 1$. Then

$$\sum_{n\leq N,}\sum_{m\leq M} a_n b_m \left(\frac{n}{m}\right) \ll (NM)^{1+\epsilon}(N^{-1/2} + M^{-1/2}).$$

Variation 1: Large Sieve Under the Hyperbola

Theorem (W., 2023)

Suppose a_n and b_m are any sequences in $\mathbb{C}$ supported on odd square-free integers such that $|a_n|, |b_n| \leq 1$. Then uniformly for all $X, z \geq 2$,

$$\sum_{\substack{z<n,m\leq X \\ nm\leq X}} \sum a_n b_m \left(\frac{n}{m}\right) \ll \frac{X(\log X)}{z^{1/4}}.$$

Variation 2: Inserting Brun Sieve Into Large Sieve

By inserting Hooley neutralisers into the large sieve we may obtain improved bounds provided we have more information on a_n and b_m. This is the key innovation of the proof.

Proposition (W.)

Let $M, N \geq W \geq 2$ *satisfy* $N \leq \frac{M}{W}$. *Suppose* a_n *and* b_m *are any sequences in* $\mathbb{C}$ *supported on odd square-free integers such that* $|a_n|, |b_n| \leq 1$. *Then:*

$$\sum_{W<n\leq N} \sum_{m\leq M} \frac{a_n b_m}{n\tau(m)} \left(\frac{n}{m}\right) \ll \frac{M(\log N)}{W^{1/2}(\log M)^{1/2}}.$$

- $1/\tau(m)$ may be replaced by appropriate multiplicative $g(m)$.

Other results

More recently I have also given more examples of thin sets occurring in Loughran–Smeets problems. Let f and g be non-singular, binary quadratic forms with discriminants $\mathrm{disc}(f), \mathrm{disc}(g) \neq \square$. Set

$$N_f(B) = \#\left\{ (u,v) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : \begin{array}{l} f(u)x^2 + v_1 v_2 y^2 = z^2 \\ \text{has a } \mathbb{Q}\text{-point} \\ \|u\| \cdot \|v\| \leq B \end{array} \right\}$$

$$N_{f,g}(B) = \#\left\{ (u,v) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : \begin{array}{l} f(u)x^2 + g(v)y^2 = z^2 \\ \text{has a } \mathbb{Q}\text{-point} \\ \|u\| \cdot \|v\| \leq B \end{array} \right\}.$$

Other results

Theorem (W., (2024))

Let f and g be non-singular, binary quadratic forms with discriminants $\mathrm{disc}(f), \mathrm{disc}(g) \neq \square$. Suppose also that g represents a square number. Then

$$B^2 \ll N_f(B) \ll B^2$$
$$B^2 \ll N_{f,g}(B) \ll B^2 \log\log B$$

Other results

If Δ_f and $\Delta_{f,g}$ are the delta invariants for these problems, then we have

$$\Delta_f = 3/2 \text{ and } \Delta_{f,g} = 1.$$

The lower bounds in the previous theorem are a result of the thin sets,

$$\{(u,v) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : v_1 v_2 = \square\}$$
$$\{(u,v) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : g(u_1, u_2) = \square\}$$

Future Work: What happens outside of these thin sets?

Sketch of proof

For each prime p, we find a set $\Omega_p \subseteq \left(\mathbb{Z}/p^2\mathbb{Z}\right)^4$, such that the conic fibre over $(u, v) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ fails to have a $\mathbb{Q}_p$ point if and only if $(u_1, u_2, v_1, v_2) \bmod p^2 \in \Omega_p$. We then define $N^*(T_1, T_2, S_1, S_2)$

$$\#\left\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^2_{prim} \times \mathbb{Z}^2_{prim} : \begin{array}{l} T_i \leq u_i \leq 2T_i, \\ S_i \leq v_i \leq 2S_i \; \forall i \in \{0, 1\} \\ (\mathbf{u}, \mathbf{v}) \bmod p^2 \notin \Omega_p \\ \forall p \leq R^{1/100} \end{array} \right\}$$

for $R = \min(T_1, T_2, S_1, S_2)$.

Sketch of proof

We may then write

$$N_{f,g}(B) \ll \sum_{T_0,T_1,S_0,S_1} N^*(T_1, T_2, S_1, S_2)$$

where the T_0, T_1, S_0, S_1 run over powers of 2 such that

$$1 \leq \max\{T_0, T_1\} \cdot \max\{S_0, S_1\} \leq B.$$

A note on the large sieve

To prove the previous result we use the following large sieve,

Lemma

Let

$$X = \{\mathbf{n} \in \mathbb{Z}^r : M_j \le n_j \le M_j + N_j, \ \text{for } 1 \le j \le r\}$$

and suppose $Y_p \subseteq (\mathbb{Z}/p^s\mathbb{Z})^r$ *for all* p *where* $s \in$ *is fixed. Then*

$$\{\mathbf{n} \in X : \mathbf{n} \bmod p^s \notin Y_p, \ \text{for all } p \le L\} \ll \frac{\prod_{i=1}^r \left(\sqrt{N_j} + L^s\right)^2}{F(L)}$$

where $F(L) = \sum_{m \le L} \mu^2(m) \prod_{p|m} \left(\frac{|Y_p|}{p^{rs} - |Y_p|}\right)$.

The combined application of skewed boxes and $\mathbb{Z}/p^s\mathbb{Z}$ congruence classes is a new feature. Nevertheless, this is still a consequence of classical large sieve results.

Sketch of proof

Applying this to each $N^*(T_0, T_1, S_0, S_1)$ we obtain

$$N_{f,g}(B) \ll \sum_{T_0, T_1, S_0, S_1} \frac{T_0 T_1 S_0 S_1}{(1 + \log R)^{\Delta_{f,g}}}$$

which leads (eventually) to

$$N_{f,g}(B) \ll B^2 \sum_{0 \leq j \leq \log B} \frac{1}{(1+j)^{\Delta_{f,g}}}.$$

So if $\Delta_{f,g} = 1$...

Some questions

(1) Can we find any other local solubility problems where a $\log\log B$ appears (perhaps using a more general biprojective variety as a base)?

(2) Is there a geometric explanation for this phenomenon?

(3) What about thin sets occurring purely in the Loughran–Smeets setting - can we understand when they occur?