Ranks in elliptic fibrations

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(Jointly with Peter Koymans & Carlo Pagano)

DRP2025

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- How to detect **disasters**?

  For positive integer t define: $$r_t = \begin{cases} 1, & t \text{ odd} \\ 0, & t \text{ even} \neq 2^a \\ \log t, & t = 2^a. \end{cases}$$

  The average of r_t is $1/2$.

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  The average of 3^{r_t} is ∞.

`Moments are better at seeing disasters`

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- applications to Cohen–Lenstra

Theorem (KPS, 2024) disasters are rare

Fix any $N \in (0, \infty)$ and arbitrary $P \neq 0$ in $\mathbb{Z}[t]$. Then $\exists c > 0$:

$$\frac{1}{X} \sum_{\substack{t \in \mathbb{Z}, |t| \le X \\ P(t) \neq 0}} N^{\mathrm{rank}(P(t)y^2 = x^3 - x)} \le c.$$

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- Fouvry 1991 $y^2 = x^3 + B$ (GRH + BSD) $N \leq \sqrt{3}$
- Heath-Brown 1993 for linear $P(t)$.
- Fouvry–Pomykala 1994 $y^2 = x^3 + A(t)x + B(t)$ (GRH + BSD)
- Michel 1995: familes of abelian varieties (GRH + BSD)

Rank jumps

Silverman 1985 conjectured

$$\mathrm{rank} E(\mathbb{A}^1) \leq \mathrm{rank} E(t) \leq 1 + \mathrm{rank} E(\mathbb{A}^1)$$

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`jumps typically bounded`

Corollary: Fix any $n > 1$. In our families

$$\mathrm{rank}\, E(t) \leq \log n \quad \text{with probability} \quad > 1 - \frac{1}{n}.$$

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- same with $d > 0$
- $n = 2$ Gauss,
- $n = 3$ Davenport-Heilbronn 1972,
- $n = 4$ Fouvry-Klüners 2007,
- $n = 2^k$ Smith 2017,
- $n = 6$ Chan-Koymans-Pagano-S. 2023 (order of magnitude)

Theorem (KPS 2024) h_3, h_{2^k} statistically independent

Fix $k \in \mathbb{N}$. There are $c, c' > 0$ such that

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- $h_{3\cdot 2^k} = h_3 h_{2^k}$ but $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$
- level of distribution for h_a & moments for $h_b \Rightarrow$ Average of h_{ab}.

Proof

- Proof different than h_6 (no easy interpretation when $k \geq 2$.)
- h_{2^k} can be controlled by a matrix with quadratic symbols.
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Proofs follow similar paths, talk on h_{12}

Tool 1: description of h_3

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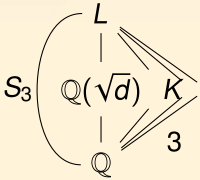

S_3-extensions contain 3 cubic fields K

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Belabas (2010) and Bhargava–Taniguchi–Thorne (2024) proved:

Theorem

For all $X \geq 1$ and $q \leq X^{1/10}$ we have

$$\sum_{\substack{0<d\leq X \\ d\equiv 0 \pmod q}} (h_3(-d)-1) = \frac{3}{\pi^2} h(q) X + O(X^{6/7}),$$

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- Asymptotic for most moduli q up to a power of X

`level of distribution`

Tool 2: bounding h_{2^k}

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- for negative discriminants: $\frac{h_4}{h_2} = 2^{\mathrm{rk}_4 \mathrm{Cl}(\mathbb{Q}(\sqrt{d}))}$

Theorem (Rédei)

Let $d < 0$ odd discriminant with prime divisors $p_1 < p_2 < \ldots < p_\omega$.

Then $\boxed{\mathrm{rk}_4 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) = \omega - 1 - \mathrm{rk}\mathcal{R}(d)}$, where

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$$\mathcal{R}(d) = \begin{bmatrix} *_1 & (\frac{p_i'}{p_1}) & \\ & \ddots & \\ & (\frac{p_i'}{p_\omega}) & *_\omega \end{bmatrix} \in \mathbb{F}_2^{\omega\times\omega}, \quad *_1 = \prod_{i\neq 1} \left(\frac{p_i'}{p_1}\right), \quad p' = \left(\frac{-1}{p}\right)p.$$

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- $h_{3\cdot 2^k} \ll h_3 2^{\omega k} 2^{-(k-1)\mathrm{rank}\mathcal{R}}$ gives too many logarithms.

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Extra property: $g(d,n) = g(d, n+d)$.

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- Nair–Tenenbaum (1992) generalised Erdős to submultiplicative divisor-bounded functions
- Crucial in Manin's conjecture for surfaces.

Tool 3: averages of functions

Theorem (Wolke 1971, Chan–Koymans–Pagano–S 2023)

For any $w_n \geq 0$ and an integer sequence $a_n = O(n^{2025})$ assume

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for some “nice” h.

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$$\sum_{\substack{1 \leq n \leq X \\ q | a_n}} w_n = h(q)X + O(X^{0.999}) \ \ \forall q \leq X^{0.001},$$

for some "nice" h. For any submultiplicative divisor-bounded f,

$$\sum_{1 \leq n \leq X} w_n f(a_n) \ll X \prod_{p \leq X} (1 - h(p)) \sum_{1 \leq n \leq X} f(n) h(n).$$

- Theorem (Sieve Result: CKP 2024)
  Can replace submultiplicative divisor-bounded by

$$\boxed{f(nm) \leq g(n,m) d(m)^c} \ \ \forall \gcd(n,m) = 1$$

where $g : \mathbb{N}^2 \to [0, \infty)$ satisfies $g(n,m) = g(n, m+n)$.

Conclusion of proof

Tool 1 — $h_3 - 1$ has level of distribution

Tool 2 — Explicit description of h_4 via Rédei twists

Tool 3: Sieve Result

$$\sum_{n \le X} \underbrace{(h_3 - 1)}_{w_n} \underbrace{h_4}_{f} \ll \frac{X}{\log X} \sum_{n \le X} \frac{h_4(n)}{n}$$

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- Character-sum method of Heath-Brown and Fouvry–Klüners

$$\sum_{d \le X} f(d) h_4(d), \quad f \text{ multiplicative.}$$

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Thank You!

Questions?