The rationality problem for real quadric surface families over the projective line

joint work with Alena Pirutka

Jean-Louis Colliot-Thélène
CNRS et Université Paris-Saclay

Diophantine and Rationality Problems (DRP2025)
10.03.25 to 14.03.2025
Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences

Basic rational varieties over an algebraically closed field : quadrics ; total space of a family of positive dimensional quadrics over the projective line.

A basic birational invariant of smooth, projective, geometrically connected varieties X over the field $\mathbb{R}$: the number of connected components of the topological space $X(\mathbb{R})$.

Beniamino Segre (1951) : a smooth real cubic surface $X \subset \mathbb{P}^3_{\mathbb{R}}$ is $\mathbb{R}$-unirational, is $\mathbb{C}$-rational, but if $X(\mathbb{R})$ has two connected components, it is not $\mathbb{R}$-rational.

A smooth, projective, geometrically rational surface X over $\mathbb{R}$ is $\mathbb{R}$-rational if and only if $X(\mathbb{R})$ is nonempty and connected (Comessatti 1913, Silhol 1989). Proof uses birational classification.

Suppose $X(\mathbb{R})$ nonempty and connected and X is rational over $\mathbb{C}$.

Is X $\mathbb{R}$-rational ?

(Weaker) Is X stably $\mathbb{R}$-rational ?

(Weaker) Is X universally Chow-trivial : for any overfield $F/\mathbb{R}$, is the degree map $deg_F : CH_0(X_F) \to \mathbb{Z}$ an isomorphism ?
(Equivalently, there is a "decomposition of the diagonal" for $X/\mathbb{R}$)
Enough to prove it for $F = \mathbb{R}(X)$.

(Weaker) For any field $F/\mathbb{R}$, are the maps
$H^i(F, \mathbb{Z}/2) \to H^i_{nr}(F(X)/F, \mathbb{Z}/2)$ isomorphisms ?
Is the map $\mathrm{Br}(F) \to \mathrm{Br}(X_F)$ an isomoprhism ?

Over any field k, the question of rationality over k of geometrically rational threefolds X/k has been the topic of works of Hassett-Tschinkel and Benoist-Wittenberg. The IJT (intermediate Jacobian torsor) method is an extension to arbitrary ground fields of the method used by Clemens-Griffiths to disprove rationality of cubic threefolds over $\mathbb{C}$.
Let us consider $X = X_{2,2} \subset \mathbb{P}^5_{\mathbb{R}}$ a smooth complete intersection of two quadrics. Over $\mathbb{C}$, rational. Over $\mathbb{R}$, the IJT method gives non-$\mathbb{R}$-rationality of X when X contains no real line (may happen with $X(\mathbb{R})$ connected). It does not give information on stable rationality or universal CH_0-triviality.

Let $X \to \mathbb{P}^1_{\mathbb{R}}$ be a family of quadrics of relative dimension $d \geq 1$. with smooth total space $X/\mathbb{R}$.

$X_{\mathbb{C}}$ is rational. If $X(\mathbb{R})$ is nonempty, X is $\mathbb{R}$-unirational (exercise).

If $X(\mathbb{R})$ is connected, is X rational over $\mathbb{R}$? Is it at least stably rational ? Is it universally CH_0-trivial ?

For $d = 1$, yes for all questions. We shall consider the case $d = 2$, quadric surface fibrations.

First negative answer, $d = 2$

Let X be a smooth projective model of the variety given by the equation

$$x^2 + (1+u^2)y^2 - u(z^2+t^2) = 0$$

in $\mathbb{P}^3 \times \mathbb{A}^1$, coordinates $(x, y, z, t; u)$.

Real points cover exactly $u \geq 0$, real fibres are connected. This gives connectedness of the real locus. Over $u = 0$ and over $u = \infty$ the family degenerates to two conjugate planes. One computes $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{R}) \neq 0$. The class $(-1, u) \in \mathrm{Br}(\mathbb{R}(X))$ is not in the image of $\mathrm{Br}(\mathbb{R})$ and is unramified.

This implies that X is not universally CH_0-trivial, hence not $\mathbb{R}$-rational and not even stably $\mathbb{R}$-rational.

Definition (for this talk).
Let k be a field of char. 0. Let $X \to \mathbb{P}^1_k$ be a quadric surface fibration. We call it a **good fibration** if X/k is smooth projective and all the geometric fibres of $X \to \mathbb{P}^1_k$ are geometrically integral.

In other words, the fibres are smooth or cones over a smooth conic.

The previous example is not a good fibration.

For good fibrations, $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is onto.

A reminder on quadric surface fibrations

Given a field k, say of char. zero, and a quadric surface fibration $X \to \mathbb{P}^1_k$ the generic fibre is a smooth quadric $Y \subset \mathbb{P}^3_K$ where $K = k(\mathbb{P}^1)$. Let q/K be a 4-dimensional quadratic form defining Y. Let L/K be the quadratic extension defined by the discriminant of q.
Thus $L = k(\Delta)$ for Δ a smooth hyperelliptic curve over k. For any smooth plane (conic) section Z/K of $Y \subset \mathbb{P}^3_K$, the image of the associated quaternion class in $\mathrm{Br}(K)$ under $\mathrm{Br}(K) \to \mathrm{Br}(L)$ is a well defined element $\beta \in \mathrm{Br}(L)$.
If $X/\mathbb{P}^1_k$ is a good fibration, we have $\beta \in \mathrm{Br}(\Delta) \subset \mathrm{Br}(L)$.

Back to the problem : Second negative answer, $d = 2$

What about good fibrations ?

In the 2021 paper by Hassett and Tschinkel, one finds examples of smooth $Y_{2,2} \subset \mathbb{P}^5_{\mathbb{R}}$ with $Y(\mathbb{R})$ connected and Y not $\mathbb{R}$-rational because of IJT obstruction.
One may birationally transform this into a good quadric surface fibration $X \to \mathbb{P}^1_{\mathbb{R}}$ with 6 geometric singular fibres. Here the quotient $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{R}) = 0$, $X(\mathbb{R})$ is connected and X is not rational over $\mathbb{R}$.

Under the **additional assumption :**
The class of $\beta \in \mathrm{Br}(\Delta) \subset \mathrm{Br}(k(\Delta))$ *is* **not** *in the image of* $\mathrm{Br}(k) \to \mathrm{Br}(k(\Delta))$

Wittenberg (2023) applies the IJT method to good quadric surface fibrations $X/\mathbb{P}^1_k$ with at least 6 geometric singular fibres, with application to the problem over the reals.
For instance $X/\mathbb{P}^1_{\mathbb{R}}$ given by the affine equation

$$(u^2-1)x^2+(u^2-2)y^2+(u^2-3)z^2=1$$

is not rational over $\mathbb{R}$ but $X(\mathbb{R})$ is connected.
The IJT method does not work without the *additional assumption*.

The following problems remain open :

Suppose $p : X \to \mathbb{P}^1_{\mathbb{R}}$ is a *good quadric surface fibration*. Suppose $X(\mathbb{R})$ is connected.

(1) Is $X/\mathbb{R}$ stably rational ? Is it universally CH_0-trivial ?

(2) If the *additional assumption* fails, i.e. if $\beta \in \mathrm{Br}(\Delta)$ is in the image of $\mathrm{Br}(\mathbb{R})$, is X rational over $\mathbb{R}$?

In this talk, we shall concentrate on a very concrete case.

Let $p(u) \in \mathbb{R}[u]$ a monic, nonconstant, even degree, separable, polynomial, strictly positive on $\mathbb{R}$. We shall consider **special quadric surface fibrations** $X/\mathbb{P}^1_{\mathbb{R}}$, given by an affine model

$$x^2 + y^2 + z^2 = u.p(u),$$

with the projection to $\mathbb{P}^1$ given by the u coordinate.
The space $X(\mathbb{R})$ is connected. The curve Δ is given by $w^2 = v.p(-v)$. *The additional assumption fails*. Indeed β is given by $(-1,-1)$. So the IJT method may not be used.

Is X rational over $\mathbb{R}$? is X stably rational over $\mathbb{R}$? Entirely open !

MAIN CONCRETE RESULT : **For** $p(u) = u^2 + au + b$ **positive, we shall give conditions in terms of the** j**-invariant of the elliptic curve** $w^2 = v.p(-v)$ **which imply that** X **is universally** CH_0**-trivial.**

This will rely on a general criterion.

Theorem *Let $p(u) \in \mathbb{R}[u]$ a monic, nonconstant, even degree, separable, polynomial, strictly positive on $\mathbb{R}$. Let $X/\mathbb{P}^1_{\mathbb{R}}$ be a special quadric fibration with affine equation*

$$x^2 + y^2 + z^2 = u.p(u).$$

Let $\Delta/\mathbb{R}$ be the smooth projective curve with affine equation

$$w^2 = v.p(-v).$$

Let $W/\mathbb{R}$ the fourfold given by $W := X \times_{\mathbb{R}} \Delta$. The following conditions are equivalent :

- *The variety X is universally CH_0-trivial.*
- *The rational function $u + v \in \mathbb{R}(W)$ is a sum of 4 squares.*
- *The cup-product $(u + v, -1, -1) \in H^3(\mathbb{R}(W), \mathbb{Z}/2)$ vanishes*
- *For $F = \mathbb{R}(\Delta)$, the map $H^3(F, \mathbb{Z}/2) \to H^3_{nr}(F(X)/F, \mathbb{Z}/2)$ is an isomorphism.*

Fact : $u + v$ is a sum of 6 squares in $\mathbb{R}(W)$.

Ingredients in the proof of the theorem.

Results from the previous century on algebraic K-theory and quadratic forms (Arason, Merkurjev, Suslin, Kahn-Rost-Sujatha).

Study of Chow groups of zero-cycles for threefolds fibred into quadric surfaces over the projective line
[CTSk] CT-Skorobogatov, J. K-Theory 7 (1993).

Let $A_0(X) \subset CH_0(X)$ the group of degree zero cycle classes. Let k be a field. Let $X/\mathbb{P}^1_k$ be a good nonconstant quadric surface fibration $X/\mathbb{P}^1_k$. Let the double cover $\Delta/\mathbb{P}^1_k$ and $\beta \in \mathrm{Br}(\Delta)[2] \subset H^2(k(\Delta), \mathbb{Z}/2)$ as above.

[CTSk] gives an **injection**
$\Phi : A_0(X) \hookrightarrow H^3_{nr}(k(\Delta)/k, \mathbb{Z}/2)/[H^1(k, \mathbb{Z}/2) \cup (\beta)]$.

Let $X \to \mathbb{P}^1_{\mathbb{R}}$ be a special quadric fibration. Here $\Delta(\mathbb{R}) \neq \emptyset$ and $\beta = (-1, -1)_{\mathbb{R}}$ (the *additional assumption* fails). Let
$W = X \times_{\mathbb{R}} \Delta$.
Using precisely this, looking at $k = \mathbb{R}(X)$ and $X \times_{\mathbb{R}} \mathbb{R}(X)$, using [CTSk], we prove : *the image under Φ of the difference between the generic point of X and an $\mathbb{R}$-rational point above $u = \infty$ vanishes (that is, $X/\mathbb{R}$ is universally CH_0-trivial, that is, there is a decomposition of the diagonal for $X/\mathbb{R}$) if and only if $(u + v, -1, -1) = 0 \in H^3(\mathbb{R}(W), \mathbb{Z}/2)$, if and only if $u + v$ is a sum of 4 squares in $\mathbb{R}(W)$.*

CH_0-triviality via sums of 4 squares, $p(u)$ of degree 2

Theorem. *Let $p(u) = u^2 + au + b \in \mathbb{R}[u]$ be separable and nonnegative. Let $X/\mathbb{P}^1_{\mathbb{R}}$ be a special quadric fibration with affine equation*

$$x^2 + y^2 + z^2 = u.p(u).$$

If $b \geq a^2/3$, then X is universally CH_0-trivial.

This covers the case $p(u) = u.(u^2 + 1)$ but does not cover the range $a^2/3 > b > a^2/4$.

Proof. Recall that $W = X \times_{\mathbb{R}} \Delta$, that X is defined by $x^2 + y^2 + z^2 = u.p(u)$ and Δ by $w^2 = v.p(-v)$. Thus *$up(u) + vp(-v)$ is a sum of 4 squares in $\mathbb{R}(W)$.*

We have the identity

$$up(u) + vp(-v) = (u+v)(u^2 - uv + v^2 + au - av + b) =$$

$$= (u+v)\left(\left(u + \frac{a-v}{2}\right)^2 + \frac{3}{4}\left(v - \frac{a}{3}\right)^2 + b - \frac{a^2}{3}\right).$$

If $b - \frac{a^2}{3} \geq 0$, then $\frac{up(u)+vp(-v)}{u+v}$ is a sum of 3 squares in $\mathbb{R}(u,v)$, hence a sum of 4 squares.

Hence if $b \geq \frac{a^2}{3}$, then $u + v \in \mathbb{R}(W)$ is the quotient of sum of 4 squares by a sum of 4 squares, hence is a sum of four squares.

Universal Chow triviality for $deg(p(u)) = 2$ via unramified H^3 :

Hypotheses on $p(u)$ ensuring that for $F = \mathbb{R}(\Delta)$, the map $H^3(F, \mathbb{Z}/2) \to H^3_{nr}(F(X)/F, \mathbb{Z}/2)$ is an isomorphism.

Theorem. *Let $p(u) \in \mathbb{R}[u]$ be a positive separable polynomial of degree 2. Let $X/\mathbb{P}^1_{\mathbb{R}}$ be a special quadric fibration with affine equation*

$$x^2 + y^2 + z^2 = u.p(u).$$

Assume that the elliptic curve $E/\mathbb{R}$ defined by $z^2 = u.p(u)$ has "odd" complex multiplication, namely $End_{\mathbb{C}} E = \mathbb{Z}[\omega]$, with $\omega^2 - d\omega + c = 0$, $c, d \in \mathbb{Z}$ and d odd. Let Δ be defined by $w^2 = v.p(-v)$. Let $F = \mathbb{R}(\Delta)$. Then the map $H^3(F, \mathbb{Z}/2) \to H^3_{nr}(F(X)/F, \mathbb{Z}/2)$ is an isomorphism, and the variety X is universally CH_0-trivial.

Main points of the proof

Let F be any overfield of $\mathbb{R}$. We consider the birational conic bundle fibration $X_F \to \mathbb{P}^2_F$ induced by the projection map

$$(x, y, z, u) \mapsto (z, u) \in \mathbb{A}^2 \subset \mathbb{P}^2.$$

The fibration is ramified along the elliptic curve $E_F \subset \mathbb{P}^2_F$ with affine equation $z^2 = u.p(u)$ and possibly along the line at infinity. By general K-theory results on conics, a standard analysis of residues and their functoriality, and use of the Bloch–Ogus complex on $\mathbb{A}^2_F$, one shows that any class $\beta \in H^3_{nr}(F(X)/F, \mathbb{Z}/2)$ trivial at an F-point is the image of a class $\alpha \in H^3(F(\mathbb{P}^2), \mathbb{Z}/2)$ whose residues aways from E_F and the line at infinity of $\mathbb{P}^2_F$ are zero, and whose residue at E_F belongs to $\mathrm{Ker}[\mathrm{Br}(E_F) \to \mathrm{Br}(E_{F'})]$, where $F' := F(\sqrt{-1})$. *We would like to get rid of this possible residue.*

Let $G = \mathbb{Z}/2 = Gal(F'/F)$. We have a standard exact sequence

$$0 \to H^2(G, F') \to \mathrm{Ker}[\mathrm{Br}(E_F) \to \mathrm{Br}(E_{F'})] \to H^1(G, \mathrm{Pic}\,(E_{F'})) \to 0.$$

Let $F = \mathbb{R}(\Delta)$. The G-lattice $M := Hom_{\mathbb{C}}(\Delta_{\mathbb{C}}, E_{\mathbb{C}})$ is isomorphic, as an abelian group, to $End_{\mathbb{C}}(E_{\mathbb{C}})$, which is abstractly $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ (last case, complex multiplication).
Using $E(\mathbb{R})$ connected, one gets $H^1(G, \mathrm{Pic}\,(E_{F'})) \simeq H^1(G, M)$.
So the obstruction to solving our problem lies in the finite group $H^1(G, M)$, which one may show is either 0 or $\mathbb{Z}/2$.

Key technical result :

Proposition. *Assume that the elliptic curve* $E/\mathbb{R}$ *defined by* $z^2 = u.p(u)$ *has "odd" complex multiplication, namely* $End_{\mathbb{C}} E = \mathbb{Z}[\omega]$, *with* $\omega^2 - d\omega + c = 0$, $c, d \in \mathbb{Z}$ *and* d *odd. Let* Δ *be defined by* $w^2 = v.p(-v)$. *Let* $F = \mathbb{R}(\Delta)$ *and* $F' = F(\sqrt{-1})$. *Then* $H^1(G, \mathrm{Pic}\,(E_{F'})) = H^1(G, M) = 0$.

Under this hypothesis, the residue of α at E_F is of the shape $(\delta, -1)$ with $\delta \in F^\times$. Over $\mathbb{A}^2_F$ the classes α and $(\delta, z^2 - up(u), -1)$ have the same residues. Their difference is thus in $H^3(F, \mathbb{Z}/2)$. Since $-(z^2 - up(u) = x^2 + y^2$ in $\mathbb{R}(X)$, the image of $(\delta, z^2 - up(u), -1)$ in $H^3(F(X), \mathbb{Z}/2)$ is $(\delta, -1, -1)$ hence comes from $H^3(F, \mathbb{Z}/2)$. So does therefore the image β of α.
QED

Comparing the two methods for $deg(p(u)) = 2$

Let $E/\mathbb{R}$ be the elliptic curve with equation $z^2 = u.(u^2 + au + b)$, $a, b \in \mathbb{R}$. We assume $b > 0$ and $0 \leq a^2/b < 4$.
One computes

$$j(E) = 256[3 - (a^2/b)]^3/[4 - (a^2/b)] \in \mathbb{R}.$$

One checks :
The invariant $j(E)$ varies in $[-\infty, 1728]$.
One has $0 \leq a^2/b \leq 3$ if and only if $0 \leq j(E) \leq 1728$.
One has $3 \leq a^2/b < 4$ if and only if $j(E) \leq 0$.
$a^2/b = 3$ corresponds to $j(E) = 0$ and $a^2/b = 0$ to $j(E) = 1728$

Chow triviality for $x^2 + y^2 + z^2 = u.(u^2 + au + b)$, with $b > 0$ and $a^2 - 4b < 0$.

The first method (sum of squares) works if and only if $j(E) \in [0, 1728]$.
We know how to make the second method (H^3 unramified) work only if E has odd complex multiplication.
If $j(E) < 0$ and E has no complex multiplication or has even complex multiplication, we have no method. In particular we have no method if $j(E)$ is negative and transcendental.

Theorem (Yu. Zarhin) : *As E varies among the real elliptic curves $z^2 = u.p(u)$ with $p(u)$ positive and odd complex multiplication, the countable set of values $j(E) \in \mathbb{R}$ is dense in $[-\infty, 1728]$.*

Examples for which we can prove X is universally CH_0-trivial

$p(u) = u^2 - 3u + 3$
E is given by $z^2 = (u-1)^3 + 1$. It has complex multiplication by ω with $\omega^2 + \omega + 1 = 0$. This it has odd CM. Here $j(E) = 0$. Both methods apply.

$p(u) = u^2 + 1$
E is given by $z^2 = u(u^2+1)$. It has $j(E) = 1728$. The first method applies. The curve E has CM by $\omega = \sqrt{-1}$, but $\omega^2 + 1 = 0$ hence it is not odd CM. The second method does not apply.

$p(u) = u^2 - 21u + 112$. Here $j(E) < 0$, the first method does not apply. The curve has complex multiplication by $\mathbb{Z}[\omega]$ with $\omega = (1+\sqrt{-7})/2$. Here $\omega^2 - \omega + 2 = 0$, thus is odd CM, the second method applies.

Open problems

Let $X/\mathbb{R}$ be a smooth projective model of the variety with affine equation $x^2 + y^2 + z^2 = u.p(u)$, with $p(u)$ monic, separable, positive on $\mathbb{R}$, of degree at least 2. Let $\Delta/\mathbb{R}$ be the curve with affine equation $w^2 = v.p(-v)$.

Are the following equivalent conditions always satisfied ?

(a) The variety $X/\mathbb{R}$ is universally CH_0-trivial.

(b) The rational function $u + v \in \mathbb{R}(X \times_{\mathbb{R}} \Delta)$ (a sum of 6 squares) is a sum of 4 squares.

- Are there examples for which X is (stably) rational over $\mathbb{R}$?
- Are there examples for which X is not (stably) rational over $\mathbb{R}$?
- Already for $deg(p) = 2$?
- What about $x^2 + y^2 + z^2 = u.(u^2 + 1)$? (which is universally CH_0-trivial).

Essentially all results we have described hold if one replaces the field $\mathbb{R}$ by an arbitrary real closed field.
But over a suitable such field, Benoist and Pirutka have found an example of a variety which is not CH_0-trivial.
Let $\mathbb{R}((t))$ be equipped with the order for which t is infinitely small positive. Let $\mathrm{R} = \cup_{n\geq 1}\mathbb{R}((t^{1/n}))$. It is a real closed field, with algebraic closure the Puiseux field.
The polynomial $p(u) = (u+1)^2 + t \in \mathrm{R}[t]$ is separable and strictly positive on R.

Theorem (O. Benoist, A. Pirutka, Dec. 2024) **The R-variety with affine equation $x^2 + y^2 + z^2 = u.p(u)$ is not CH_0-trivial, hence not stably rational.**

The field R is the limit of the fields $\mathbb{R}((t^{1/4m}))$. One lets X be the $\mathbb{R}((t))$ variety defined by $x^2 + y^2 + z^2 = u.p(u)$. One considers the curve $\Delta/\mathbb{R}((t))$ given by $w^2 = v.p(-v)$ and the product $W = X \times_{\mathbb{R}((t))} \Delta$.
They show the image of the class $\alpha = (u + v, -1, -1) \in H^3(\mathbb{R}((t))(W), \mathbb{Z}/2)$ has nontrivial image in each $H^3(\mathbb{R}((t^{1/4m}))(W), \mathbb{Z}/2)$, by computing a residue on a suitable partial model over the DVR $\mathbb{R}[[t^{1/4m}]]$. Then the image of α in $H^3(\mathrm{R}(W), \mathbb{Z}/2)$ does not vanish. The criterion from page 13 (which works over R) shows that $X/\mathbb{R}$ is not universally CH_0-trivial.

One checks that the j invariant of the curve Δ satisfies $j < 0$ and j transcendental. Hence none of the two methods we have described could have applied.