

# Counts on wild stacks

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DRP 2025, International Center for Mathematical Sciences  
March 13th, 2025  
(joint work with T. Yasuda)

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There are no definitions of heights on stacks that recover the Faltings height, if the characteristic of the base field is 2 and 3.

## Conjecture (Malle 2002)

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$$\#\{K/F - Z \mid K/F \text{ Galois and } \text{Gal}(K/F) = G, \Delta^\rho(K) < B\} \\ \sim CB^a \log(B)^{b-1},$$

for some  $C > 0$ ,  $a = a(\rho) > 0$ ,  $b = b(\rho, F) > 1$ .

# Residue map

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We denote by  $BG(K)$  the set of  $G$ -torsors over a field  $K$ . Let  $q_v$  be the cardinality of the residue field at  $v$ . There exists a canonical map  $\psi_v$ :

$$\begin{aligned}BG(F_v) &= H^1(\mathrm{Gal}(\overline{F}_v/F_v), G) \\&\rightarrow H^1(\mathrm{Gal}(\overline{F}_v/F_v^{\mathrm{un}}), G) \\&= H^1(\mathrm{Gal}(F_v^{\mathrm{tame}}/F_v^{\mathrm{un}}), G) \quad (\text{for almost all } v) \\&= \mathrm{Hom}\left(\varprojlim_{\gcd(l, q_v)=1} \mu_l, G\right) / \text{conjugation} \quad \mu_l := \text{group of } l\text{-th roots of } 1 \\&= \mathrm{Hom}(\mu_{\#G}, G) / \text{conjugation} \\&\stackrel{\mathrm{def}}{=} G(-1)(\overline{F})\end{aligned}$$

# Heuristics

There exists  $c_\rho : G(-1)(\bar{F}) \rightarrow \mathbb{R}_{\geq 0}$ , which is  $\text{Gal}(\bar{F}/F)$ -invariant, vanishing at 0, such that for almost all  $v$ , the local discriminant factorizes  $\Delta_v^\rho = c_\rho \circ \psi_v$ .

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$$\begin{aligned} Z(s) & \sim \prod_v \left( \sum_{x_v \in BG(F_v)} (\Delta_v^\rho)^{-s}(x_v) \right) \\ & = \prod_v \left( 1 + \#G(-1)_{\min}(F_v) q_v^{-\min_{x>0}(c_\rho(x)) \cdot s} + O(q_v^{-as+\varepsilon}) \right) \\ & \asymp L(s, \gamma), \end{aligned}$$

where  $\gamma$  is the corresponding Gal. rep.:

$$a := \frac{1}{\min_{x>0} c_\rho(x)}, \quad b := \#\{x \in G(-1)_{\min} / \text{Gal}(\bar{F}/F)\}.$$

# Wild case

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- 5 Gundlach, Seguin (2025),  $G$  is two-step nilpotent group, height is Artin–Schreier conductor.

The difficulty lies in the fact that there is no finite  $G(-1)$ .

## Definition

Let  $G$  be a finite (étale) group (scheme over  $\mathbb{F}_q$ ). We define

$$\Delta_G := \text{Res}_{\mathbb{F}_q((t))/\mathbb{F}_q} BG.$$

In other words

$$\Delta_G(S) = BG(S \times_{\mathbb{F}_q} \mathbb{F}_q((t))).$$

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We deduce that

$$\Delta_G(\mathbb{F}_q) = H^1(\mathbb{F}_q((t)), G) = \mathbf{G}_a(\mathbb{F}_q((t))) /_{x \mapsto x^p - x} \mathbf{G}_a(\mathbb{F}_q((t))).$$

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The element  $\mathbb{F}_q((t))[X]/X^p - X - a(t)$  identifies with

$$a(t) = a_0 + \sum_{\substack{n \in \mathbf{N} \\ p \nmid n}} a_i t^{-i}. \quad (a_0 \in B(\mathbb{Z}/p\mathbb{Z})(\mathbb{F}_q), a_i \in \mathbb{F}_q, \forall i \gg 0, a_i = 0).$$

Consider  $k = \mathbb{F}_q((t))[X]/(X^p - X - a(t)) \in \Delta_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{F}_q)$ , where  $a(t) = \sum_{\substack{i \in \mathbf{N}_0 \\ p \nmid i, \text{ if } i > 0}} a_i t^{-i}$ . Then

$$\text{cond}(k) = (-\text{ord}(a(t))) + 1 = (\max_{a_i \neq 0} i) + 1.$$

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We deduce

$$\{\text{cond} \leq n\}(\mathbb{F}_q) \cong B(\mathbb{Z}/p\mathbb{Z})(\mathbb{F}_q) \times (\mathbf{G}_a)^{n - \lfloor n/p \rfloor}(\mathbb{F}_q).$$

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$\Delta_G$  is an ind-Deligne–Mumford stack

$$B(\mathbb{Z}/p\mathbb{Z}) \times (\varinjlim_n (\mathbf{G}_a)^{n - \lfloor n/p \rfloor}) \hookrightarrow (\mathbf{G}_a)^{n+1 - \lfloor (n+1)/p \rfloor} \xrightarrow{\text{Frob}} (\mathbf{G}_a)^{n+1 - \lfloor (n+1)/p \rfloor}.$$

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$$|\{\text{cond} = n\}| \cong |(\mathbf{G}_a)^{n - \lfloor n/p \rfloor}| - |(\mathbf{G}_a)^{(n-1) - \lfloor (n-1)/p \rfloor}|$$

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## Definition

A raising function is a constructible:  $c : |\Delta_G| \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $c(0) = 0$ , which is invariant for automorphisms  $\overline{\mathbb{F}_q}((t)) \xrightarrow{\sim} \overline{\mathbb{F}_q}((t))$  induced from an automorphism  $\mathbb{F}_{q'}((t)) \xrightarrow{\sim} \mathbb{F}_q((t))$  for some  $\mathbb{F}_{q'}/\mathbb{F}_q$  finite.

# Moduli space of formal $G$ -torsors

“Residue map”:

$$\begin{aligned} BG(\mathbb{F}_q((t))) &= H^1(\mathbb{F}_q((t)), G) \rightarrow H^1(\mathbb{F}_q((t))^{\text{un}}, G) \xrightarrow{\sim} H^1(\overline{\mathbb{F}_q}((t)), G) \\ &= \Delta_G(\overline{\mathbb{F}_q}) \rightarrow |\Delta_G|. \end{aligned}$$

## Definition

For each  $v$ , fix  $F_v \cong \mathbb{F}_{q_v}((t))$ . For  $x \in BG(F)$ , define

$$H(x) = \prod_v q_v^{c(i_v(x))} \quad \text{where } i_v : BG(F) \rightarrow BG(F_v) \text{ is the canonical map.}$$

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# Heuristics

One has that

$$Z(s) = \prod_v \left( 1 + \sum_{r \in c(|\Delta_G|) - \{0\}} \#c^{-1}(r)(\mathbb{F}_{q_v}) q_v^{-rs} \right)$$

$$\stackrel{\text{Lang-Weil}}{\asymp} \prod_v \left( 1 + \sum_{r \in c(|\Delta_G|) - \{0\}} \#(\text{irr. comp. of max dim}) q_v^{\dim c^{-1}(r) - rs} \right)$$

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Under some assumptions, this has pole at

$$a(c) := \sup_{r \neq 0} \frac{1 + \dim(c^{-1}(r))}{r}$$

of order

$$b(c) := \#(\text{irr. comp. of max dim})(c^{-1}(\cup_{r_0} \{r_0\})),$$

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The heuristics is compatible with the previous results for f. ab.  $p$ -groups and cond, as well as for  $G = (\mathbb{Z}/p\mathbb{Z})^d$  and discriminant.

## Theorem (D.-Yasuda, 2025)

Let  $G$  be a finite abelian  $p$ -group. Suppose that  $c : |\Delta_G| \rightarrow \mathbf{N}_0$  is a raising function with  $a(c) < \infty$ :

- 1 there exists  $C > 0$  such that for any extension  $\mathbb{F}_{q'}/\mathbb{F}_q$  and any  $\varepsilon_0 > 0$ , one has that  $c^{-1}(r)(\mathbb{F}_{q'}) \leq C(q')^{\dim c^{-1}(r) + r\varepsilon_0}$ .
- 2 suppose  $|c(x + x_0) - c(x)|$  is bounded when  $x_0 \in W$  for some finite set  $W \subset \Delta_G(\overline{\mathbb{F}_q})$

Then for all  $\delta > 0$ ,

$$B^{a(c)} \ll \#\{x \in BG(F) \mid H(x) \leq B\} \ll B^{a(c)+\delta}.$$

The discriminant is an example.

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## Theorem (D.-Yasuda 2025)

If also  $0 < b(c) < \infty$  and  $a(c)$  is not an accumulation point of  $(1 + \dim(c^{-1}(r)))r^{-1}$ , then

$$\#\{x \in BG(F) \mid H(x) \leq B\} \asymp B^{a(c)} \log(B)^{b(c)-1}.$$

# Tame stacks

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$\text{Spec}(\mathcal{O}_K)/\mu_l \rightarrow X$ , where  $K/F_v^{\text{un}}$  is Gal. ext. with  $\text{Gal}(K/F_v^{\text{un}}) = \mu_l, l \geq 1$ ,

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The “minimal integral types” are finite in number. Restricting to closed fiber, gives a residue map:

$$\psi_v : X(F_v^{\text{un}}) \rightarrow \pi_0(\mathcal{J}_0 X), \text{ where } \mathcal{J}_0 X = \coprod_{l \geq 1} \text{Hom}^{\text{rep}}(B\mu_l, X).$$

# Heights on tame stacks

A height is defined by a pair  $(L, c)$  of a line bundle  $L$  on  $X$  and a “raising” function  $c : \pi_0(\mathcal{J}_0 X) \rightarrow \mathbb{R}_{\geq 0}$ , which is zero at the “neutral” connected component.

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## Definition

For  $x \in X(F)$ , we define

$$H_{(L,c)}(x) = H_L(x) \cdot \prod_v q_v^{c(\psi_v(i_v(x)))},$$

where  $i_v : X(F) \rightarrow X(F_v)$  is the canonical map.

## Theorem

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We assume that  $X$  is isotrivial. We extend an  $\overline{\mathbb{F}_q}((t))$ -point of a wild stack  $X$  to a representable

$$\mathrm{Spec}(\mathcal{O}_K)/G \rightarrow X, \text{ for } K/\overline{\mathbb{F}_q}((t)) \text{ Gal. ext. with } \mathrm{Gal}(K/\overline{\mathbb{F}_q}((t))) = G$$

which defines a point of the stack  $\mathcal{J}_\infty X$  of twisted formal arcs on  $X$ :

$$|\mathcal{J}_\infty X| = \coprod \mathrm{Hom}^{\mathrm{rep}}(\mathrm{Spec}(\mathcal{O}_K)/G \rightarrow X)/\text{equivalence}.$$

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$$H(x) := H_L(x) \cdot \prod_v q_v^{c(i_v(x))},$$

## Proposition

The Faltings height is defined by a pair of a line bundle and a “raising” function  $c : |\mathcal{J}_\infty X| \rightarrow \mathbb{R}$ .

Assume  $X$  is of generically trivial stabilizer. Let  $c : |\mathcal{J}_\infty X| \rightarrow \mathbb{R}_{\geq 0}$  be a raising function.

# Wild case

Assume  $X$  is of generically trivial stabilizer. Let  $c : |\mathcal{J}_\infty X| \rightarrow \mathbb{R}_{\geq 0}$  be a raising function. Let  $\bigcup A_i \subset |\mathcal{J}_\infty X|$  be a “nice” subdivision such that  $c|_{A_i}$  is constant. The tame case corresponds to  $A_i :=$  preimages for the canonical map  $\pi_0 : |\mathcal{J}_\infty X| \rightarrow |\mathcal{J}_0 X|$ .

We define  $\mathfrak{N}_X^1 = NS^1(X)_{\mathbb{R}} \oplus \bigoplus_{A_i \not\subset c=0} \mathbb{R}[A_i]$ . Each orbifold curve on  $X$ , defines a linear map on  $\mathfrak{N}_X^1$ .

# Wild case

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We define  $\mathfrak{N}_X^1 = NS^1(X)_\mathbb{R} \oplus \bigoplus_{A_i \not\subset c=0} \mathbb{R}[A_i]$ . Each orbifold curve on  $X$ , defines a linear map on  $\mathfrak{N}_X^1$ . Pseudoeffective cone could be defined *à la Boucksom-Demailly-Păun-Peternell*, that is demanding that the intersection is non-negative with *movable* orbifold curves.

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THANK YOU