On the rationality of conic bundle threefolds

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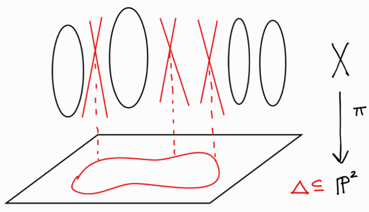

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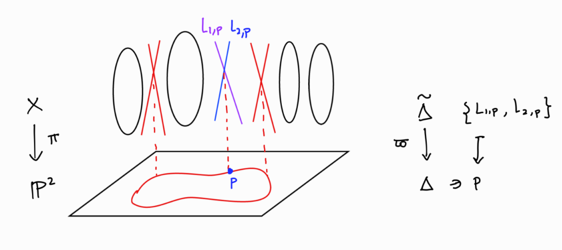

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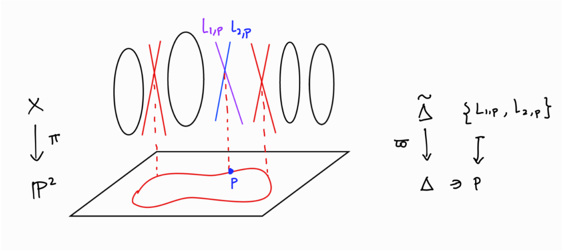

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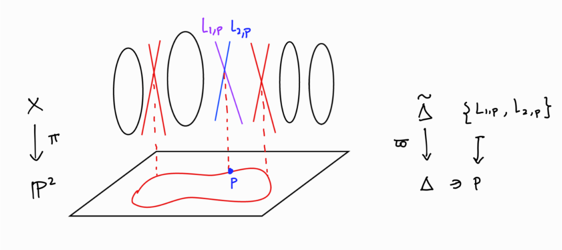

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We will assume these are smooth from now on.

Main question

Recall

A variety V is said to be *rational* over k if it has a birational map defined over k to $\mathbb{P}^N$ for some $N \in \mathbb{Z}_{\geq 0}$.

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The discriminant cover keeps track of lines in bad fibers (and some other information). So, we are asking: if you knew some finer information about lines in the non-smooth fibers of π, could you say X was rational?

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Theorem (Clemens–Griffiths, Beauville, Shokurov)

Let $X \to \mathbb{P}^2$ be a conic bundle with discriminant cover $\widetilde{\Delta}/\Delta$. Then $X \sim_{k(=\overline{k})} \mathbb{P}^3$ if and only if $\operatorname{Prym}_{\widetilde{\Delta}/\Delta}$ is isomorphic to a product of Jacobians of curves.

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Feeling

Condition on Prym might have to be replaced by condition on *torsors*, i.e. varieties that a geometrically isomorphic to the Prym.

Polarized Prym torsors

For any $m \in \mathbb{Z}_{\geq 0}$, define

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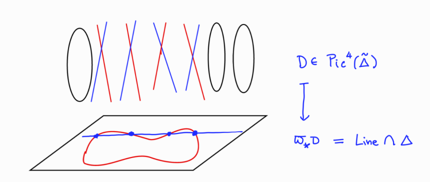

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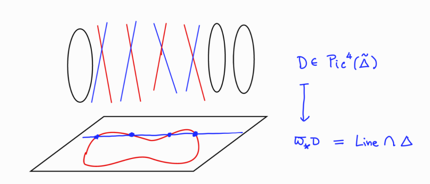

Definition

The connected components of the V_m's are called *Polarized Prym Torsors* and are indeed torsors of $\mathrm{Prym}_{\widetilde{\Delta}/\Delta}$.

Polarized Prym Obstruction

Theorem (Hassett–Tschinkel, Benoist–Wittenberg, FJSVV)

Let $X \to \mathbb{P}^2$ be a conic bundle satisfying the necessary conditions from earlier. If $X \sim_k \mathbb{P}^3$, then each Polarized Prym Torsor is isomorphic to $\mathrm{Pic}^d(\Gamma)$ for some d.

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- The PPO is a particular instance of the *Intermediate Jacobian Torsor Obstruction* that was defined by Hassett–Tschinkel and Benoist–Wittenberg for general geometrically rational threefolds.
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- Intermediate Jacobian torsors over non-algebraically closed fields look like $(\mathrm{CH}^2_{X/k})^\gamma$ for $\gamma \in \mathrm{NS}^2(X)$. So the PPO is about the structure of the space of curve classes on X.

Are we done?

Question

If you know the PPO vanishes, then is it sufficient to conclude that the conic bundle is rational?

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Let $X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched over a $(2,2)$ divisor. Then X can be written as

$$z^2 = t_0^2 Q_1(u,v,w) + 2t_0 t_1 Q_2(u,v,w) + t_1^2 Q_3(u,v,w),$$

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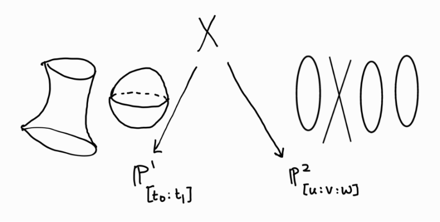

In this case $\Delta\colon Q_1 Q_3 - Q_2^2 = 0$ and Γ ($\mathrm{Prym}_{\widetilde{\Delta}/\Delta} \cong \mathrm{Jac}(\Gamma)$) is the Stein factorization of the relative Fano variety of lines on $X \to \mathbb{P}^1$.

Counterexample over $\mathbb{R}$

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\begin{aligned}
Q_1 &= -31u^2 + 12uv - 6v^2 + 9uw + 531vw + 25w^2, \\
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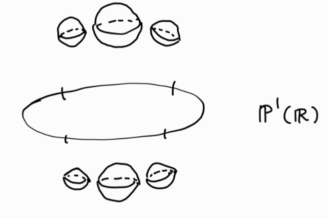

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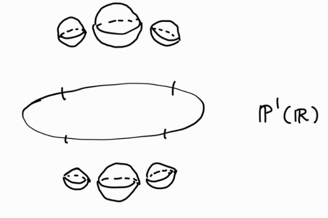

X is irrational because $X(\mathbb{R})$ is disconnected.

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Recall original question

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Purity exact sequence on 2-torsion of Brauer groups:

$$0 \to \mathrm{Br}(k)[2] \to \mathrm{Br}(k(\mathbb{P}^2))[2] \to \bigoplus_{Y \subset (\mathbb{P}^2)^{(1)}} H^1(k(Y), \mathbb{Z}/2\mathbb{Z})$$

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So $[X_{k(\mathbb{P}^2)}]$ and $[X'_{k(\mathbb{P}^2)}]$ differ by a constant Brauer class. How do we force this class to be trivial? (Imposing conditions on $\widetilde{\Delta}/\Delta$ does not seem enough)

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Theorem (FJSVV)

Let $X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a $(2,2)$ divisor with discriminant cover $\widetilde{\Delta}/\Delta$. Suppose the Polarized Prym obstruction vanishes and that $\mathrm{Prym}_{\widetilde{\Delta}/\Delta} \cong \mathrm{Jac}(\Gamma)$ with $\Gamma(k) \neq \emptyset$.

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- For some fields (e.g. finite fields, local fields), we can say more.

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- Understand the structures of the polarized Prym torsors ($m = 0, 1$ enough) in terms of lines with certain intersection theoretic properties.
- α actually comes from a conic bundle on one of these torsors.
- Show that this "agrees with" the Brauer class coming from the Fano variety of lines.

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Theorem (FJSVV)

Let $X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a $(2,2)$ divisor with discriminant cover $\widetilde{\Delta}/\Delta$. Suppose the Polarized Prym obstruction vanishes and that $\mathrm{Prym}_{\widetilde{\Delta}/\Delta} \cong \mathrm{Jac}(\Gamma)$ with $\Gamma(k) \neq \emptyset$. Then there is an $\alpha \in \mathrm{Br}(k)[2]$ such that

1. $[X_{k(\mathbb{P}^2)}] + \mathrm{im}(\alpha) = [X'_{k(\mathbb{P}^2)}]$ for a *rational* conic bundle X' with discriminant cover $\widetilde{\Delta}/\Delta$.
2. α is trivial if and only if the quadric surface bundle $X \to \mathbb{P}^1$ has a section.

- Understand the structures of the polarized Prym torsors ($m = 0, 1$ enough) in terms of lines with certain intersection theoretic properties.
- α actually comes from a conic bundle on one of these torsors.
- Show that this "agrees with" the Brauer class coming from the Fano variety of lines.

THANK YOU!