Equivariant unirationality

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joint with Cheltsov, Hassett, Kresch, Zhang

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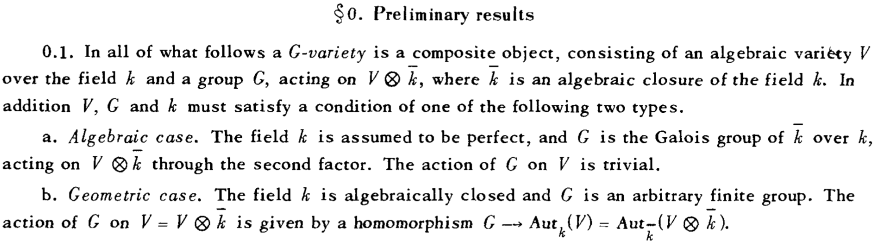
The focus today will be on birational aspects of these notions, in dimensions ≤ 3.

Manin: Rational surfaces over perfect fields I, II (1966-67)



§0. Preliminary results

0.1. In all of what follows a *G-variety* is a composite object, consisting of an algebraic variety V over the field k and a group G, acting on $V \otimes \bar{k}$, where $\bar{k}$ is an algebraic closure of the field k. In addition V, G and k must satisfy a condition of one of the following two types.

a. *Algebraic case*. The field k is assumed to be perfect, and G is the Galois group of $\bar{k}$ over k, acting on $V \otimes \bar{k}$ through the second factor. The action of G on V is trivial.

b. *Geometric case*. The field k is algebraically closed and G is an arbitrary finite group. The action of G on $V = V \otimes \bar{k}$ is given by a homomorphism $G \longrightarrow \mathrm{Aut}_k(V) = \mathrm{Aut}_{\bar{k}}(V \otimes \bar{k})$.

Geometry over nonclosed fields

Manin's motivation was to understand the interplay between geometry and arithmetic, with particular regard for geometrically rational surfaces. The main problems were:

- in arithmetic: rationality, stable rationality, unirationality, obstructions to the Hasse principle and weak approximation,

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Despite major progress on both problems over the years, there remain many open questions.

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$$X \times \mathbb{P}^m \sim \mathbb{P}^n,$$

(U) Unirationality

$$\mathbb{P}^n \dashrightarrow X.$$

In all these cases, a necessary condition is:

$$X(k) \neq \emptyset.$$

What is known?

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- (R): settled,
- (SR): open – list of possible Galois actions,
- (U): open for, e.g., del Pezzo surfaces of degree 1, and conic bundles with ≥ 8 degenerate fibers.

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Existence of G-fixed points is not a birational invariant, for nonabelian G. After passing to a suitable birational model, all stabilizers are abelian.

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The action of C_2^2 on $\mathbb{P}^1 \supset \mathbb{G}_m$ via

$$x \mapsto -x, \quad x \mapsto x^{-1}$$

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However, the $\mathfrak{D}_n$-action **is** unirational, if we allow generic stabilizers! **Exercise:** Classify unirational actions on $\mathbb{P}^1$.

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What is known?

For a rational surface satisfying Condition **(A)**, we have

- (L) settled (Pinardin-Sarikyan-Yasinsky, 2024),
- (SL) **one** open case for del Pezzo surfaces, a DP4 with a particular $C_3 \rtimes C_4$-action,
- (U) settled for generically free actions on del Pezzo surfaces of degree ≥ 3 (Duncan, 2014); open for actions with nontrivial generic stabilizers.

Other directions

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Among less popular problems in birational geometry:

- (Stable) birationality to Brauer-Severi varieties?
- (Stable) equivariant birationality to projectively linear actions?
- (Stable) birationality between geometrically rational nonrational varieties; and its equivariant analog.

Example

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Turning to X, X', with an action of $G = \mathbb{Z}/2$ fixing a genus 1 curve E, respectively E'. The Burnside formalism shows that X and X' are not G-birational if $E \neq E'$. Are X and X' stably equivariantly birational?

Plan

- Group cohomology
- Cohomological obstructions
- Unirationality of Fano threefolds

Group cohomology

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where A runs over abelian subgroups of G. This emerged in the study of Noether's problem:

Bogomolov (1987)

$$\mathrm{B}^2(G) = \mathrm{Br}_{nr}(k(V/G))$$

Group cohomology

One may also consider higher Bogomolov multipliers, e.g.,

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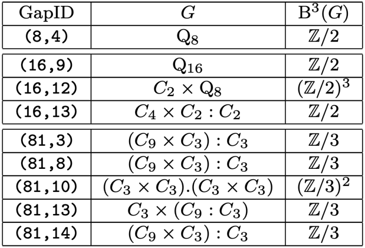
It is much easier to find a nontrivial $\mathrm{B}^3(G)$! Out of 51 groups of order 32, 23 have nontrivial $\mathrm{B}^3(G)$.



GapID	G	$\mathrm{B}^3(G)$
(8,4)	Q_8	$\mathbb{Z}/2$
(16,9)	Q_{16}	$\mathbb{Z}/2$
(16,12)	$C_2 \times \mathrm{Q}_8$	$(\mathbb{Z}/2)^3$
(16,13)	$C_4 \times C_2 : C_2$	$\mathbb{Z}/2$
(81,3)	$(C_9 \times C_3) : C_3$	$\mathbb{Z}/3$
(81,8)	$(C_9 \times C_3) : C_3$	$\mathbb{Z}/3$
(81,10)	$(C_3 \times C_3).(C_3 \times C_3)$	$(\mathbb{Z}/3)^2$
(81,13)	$C_3 \times (C_9 : C_3)$	$\mathbb{Z}/3$
(81,14)	$(C_9 \times C_3) : C_3$	$\mathbb{Z}/3$

Cohomological obstructions

Leray spectral sequence

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Let X be a smooth projective variety over $k = \bar{k}$ with a regular action by a finite group G. We have the basic exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(G, k^\times) \to \mathrm{Pic}(X, G) \to \mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G, k^\times) \\ \xrightarrow{\gamma} \mathrm{Br}([X/G]) \xrightarrow{\beta} \mathrm{H}^1(G, \mathrm{Pic}(X)) \xrightarrow{\delta_3} \mathrm{H}^3(G, k^\times), \end{aligned}$$

where

- $\mathrm{Pic}(X)$ is the Picard group, a G-module,
- $[X/G]$ is the quotient stack, and
- $\mathrm{Pic}(X, G) = \mathrm{Pic}([X/G])$ is the group of isomorphism classes of G-linearized line bundles on X.

Brauer group

A well-studied invariant is

$$\mathrm{H}^1(G, \mathrm{Pic}(X)).$$

All possibilities (with normal forms) have been computed, by Swinnerton-Dyer, Manin, Kunyavski-Skorobogatov-Tsfasman, . . ., Urabe. In particular, this group vanishes for all del Pezzo surfaces of degree ≥ 5.

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A necessary condition for (stable) linearizability is

$$\mathrm{H}^1(H, \mathrm{Pic}(X)) = 0, \quad \forall H \subseteq G.$$

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For del Pezzo surfaces, there are 32 types over *k* (T.-Kaiqi Yang 2020) and 1 type over *BG* (Prokhorov 2013).

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All possibilities, for actions of abelian G on del Pezzo surfaces and on certain conic bundles over $\mathbb{P}^2$, have been computed by Pirutka-Zhang (2024).

Amitsur groups

The basic exact sequence

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also gives rise to the following stable birational invariants:

- $\mathrm{Am}^2(X, G) := \mathrm{Im}(\delta_2)$, the *Amitsur group*, and
- $\mathrm{Am}^3(X, G) := \mathrm{Im}(\delta_3)$.

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We have, for $j = 2, 3$:

- If X satisfies Condition **(A)** then $\mathrm{Am}^j(X, G) \subseteq \mathrm{B}^j(G)$.
- If X is G-unirational then $\mathrm{Am}^j(X, G) = 0$.

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$$\mathrm{Am}^2(X, G) = 0,$$

for minimal del Pezzo surfaces of degree ≤ 6.

Amitsur groups – T.-Zhang (2025)

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Then X is one of the following:

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Then X is one of the following:

- The Fermat cubic surface with an action of $H = C_3^3$.
- A DP2 with actions of

$$H = C_2^2 \times C_4, \quad \mathfrak{D}_4 \rtimes C_2, \quad C_4 \wr C_2.$$

- or ...

Amitsur groups – T.-Zhang (2025)

A DP2, given by

$$w^2 = x_1^4 + x_2^4 + x_3^4 + a x_1^2 x_2^2, \quad a \in k,$$

with an action of $G = \mathrm{Q}_8$

$$\begin{aligned}(x_1, x_2, x_3, w) &\mapsto (\zeta_4^3 x_1, \zeta_4 x_2, x_3, w)\\ (x_1, x_2, x_3, w) &\mapsto (\zeta_4 x_2, \zeta_4 x_1, -x_3, -w).\end{aligned}$$

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The stabilizer stratification is

	Strata	Stabilizer	Residual action	dim	deg
1–4	$\mathfrak{p}_i$	C_4	triv	0	1
5	$\mathfrak{p}_5$	C_4	triv	0	1
6	E	C_2	C_2^2	1	4

Amitsur groups – T.-Zhang (2025)

The (smooth) curve $E = \{x_3 = 0\} \cap X$ has generic stabilizer C_2 and carries a residual action of $\mathrm{Q}_8/C_2 \simeq C_2^2$, we have $\mathfrak{p}_1, \ldots, \mathfrak{p}_4 \in E$ and $\mathfrak{p}_5 \notin E$.

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$$\mathrm{Br}([X/G]) = 0.$$

On the other hand,

$$\mathrm{H}^2(G, k^\times) = 0, \quad \mathrm{H}^1(G, \mathrm{Pic}(X)) = \mathbb{Z}/2.$$

It follows that

$$\mathrm{Am}^3(X, G) \neq 0,$$

in particular, X is not (projectively) G-unirational.

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This is the only action on del Pezzo surfaces with Condition **(A)** and nontrivial Amitsur obstruction.

Amitsur groups – T.-Zhang (2025)

The (smooth) curve $E = \{x_3 = 0\} \cap X$ has generic stabilizer C_2 and carries a residual action of $\mathrm{Q}_8/C_2 \simeq C_2^2$, we have $\mathfrak{p}_1, \ldots, \mathfrak{p}_4 \in E$ and $\mathfrak{p}_5 \notin E$. We have $g(E) = 1, g(E/G) = 0$, and

$$\mathrm{Br}([X/G]) = 0.$$

On the other hand,

$$\mathrm{H}^2(G, k^\times) = 0, \quad \mathrm{H}^1(G, \mathrm{Pic}(X)) = \mathbb{Z}/2.$$

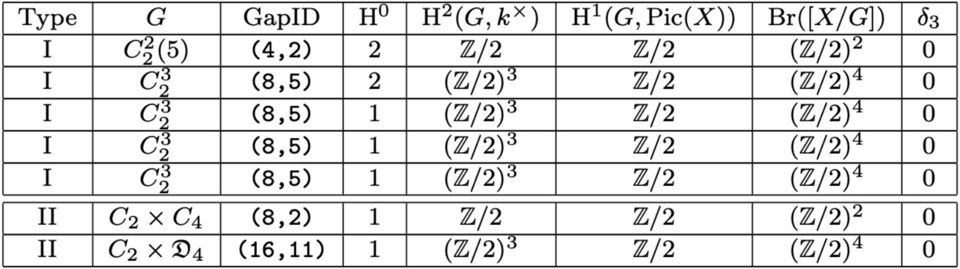
It follows that

$$\mathrm{Am}^3(X, G) \neq 0,$$

in particular, X is not (projectively) G-unirational.

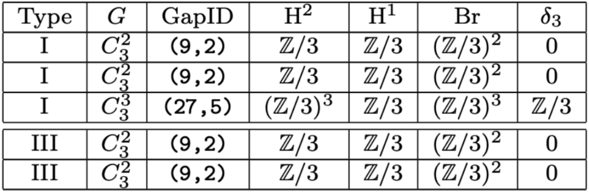
This is the only action on del Pezzo surfaces with Condition **(A)** and nontrivial Amitsur obstruction. Recall that $\mathrm{B}^3(G) = \mathbb{Z}/2\mathbb{Z}$.

Tables: DP4



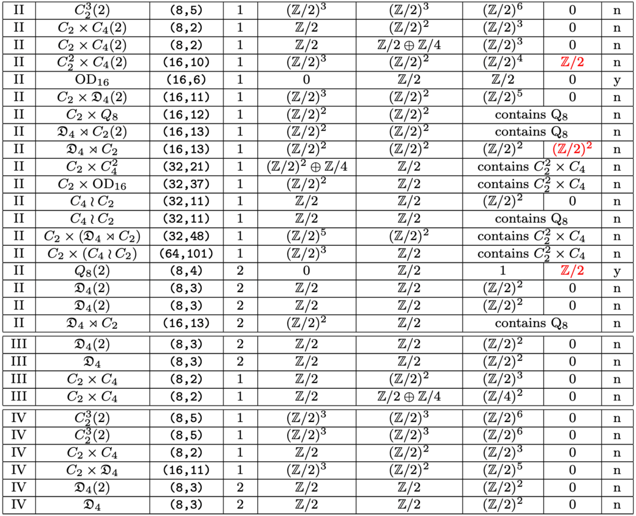
Type	G	GapID	H^0	$\mathrm{H}^2(G, k^\times)$	$\mathrm{H}^1(G, \mathrm{Pic}(X))$	$\mathrm{Br}([X/G])$	δ_3
I	$C_2^2(5)$	`(4,2)`	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
I	C_2^3	`(8,5)`	2	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	0
I	C_2^3	`(8,5)`	1	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	0
I	C_2^3	`(8,5)`	1	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	0
I	C_2^3	`(8,5)`	1	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	0
II	$C_2 \times C_4$	`(8,2)`	1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
II	$C_2 \times \mathfrak{D}_4$	`(16,11)`	1	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	0

Tables: DP3



Type	G	GapID	H^2	H^1	Br	δ_3
I	C_3^2	`(9,2)`	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$(\mathbb{Z}/3)^2$	0
I	C_3^2	`(9,2)`	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$(\mathbb{Z}/3)^2$	0
I	C_3^3	`(27,5)`	$(\mathbb{Z}/3)^3$	$\mathbb{Z}/3$	$(\mathbb{Z}/3)^3$	$\mathbb{Z}/3$
III	C_3^2	`(9,2)`	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$(\mathbb{Z}/3)^2$	0
III	C_3^2	`(9,2)`	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$(\mathbb{Z}/3)^2$	0

Tables: DP2



II	$C_2^3(2)$	(8,5)	1	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^6$	0	n
II	$C_2 \times C_4(2)$	(8,2)	1	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	0	n
II	$C_2 \times C_4(2)$	(8,2)	1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^3$	0	n
II	$C_2^2 \times C_4(2)$	(16,10)	1	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	$\mathbb{Z}/2$	n
II	OD_{16}	(16,6)	1	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	y
II	$C_2 \times \mathfrak{D}_4(2)$	(16,11)	1	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^5$	0	n
II	$C_2 \times Q_8$	(16,12)	1	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	contains Q_8		n
II	$\mathfrak{D}_4 \rtimes C_2(2)$	(16,13)	1	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	contains Q_8		n
II	$\mathfrak{D}_4 \rtimes C_2$	(16,13)	1	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	n
II	$C_2 \times C_4^2$	(32,21)	1	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	contains $C_2^2 \times C_4$		n
II	$C_2 \times \mathrm{OD}_{16}$	(32,37)	1	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	contains $C_2^2 \times C_4$		n
II	$C_4 \wr C_2$	(32,11)	1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	n
II	$C_4 \wr C_2$	(32,11)	1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	contains Q_8		n
II	$C_2 \times (\mathfrak{D}_4 \rtimes C_2)$	(32,48)	1	$(\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^2$	contains $C_2^2 \times C_4$		n
II	$C_2 \times (C_4 \wr C_2)$	(64,101)	1	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	contains $C_2^2 \times C_4$		n
II	$Q_8(2)$	(8,4)	2	0	$\mathbb{Z}/2$	1	$\mathbb{Z}/2$	y
II	$\mathfrak{D}_4(2)$	(8,3)	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	n
II	$\mathfrak{D}_4(2)$	(8,3)	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	n
II	$\mathfrak{D}_4 \rtimes C_2$	(16,13)	2	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	contains Q_8		n
III	$\mathfrak{D}_4(2)$	(8,3)	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	n
III	$\mathfrak{D}_4$	(8,3)	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	n
III	$C_2 \times C_4$	(8,2)	1	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	0	n
III	$C_2 \times C_4$	(8,2)	1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$(\mathbb{Z}/4)^2$	0	n
IV	$C_2^3(2)$	(8,5)	1	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^6$	0	n
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Linearizability / projective linearizability

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One may also consider projective unirationality, i.e., being dominated by $\mathbb{P}(V)$, where V is a faithful representation of a central extension of G. In practice, it is difficult to construct such dominant rational maps.

Condition (A)

Let G be a finite group with nontrivial $\mathrm{B}^2(G)$. Fix a nonzero

$$\alpha \in \mathrm{B}^2(G) \subset \mathrm{H}^2(G, k^\times),$$

and consider a central extension

$$1 \to Z \to \tilde{G} \to G \to 1$$

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representing α. Choosing a faithful representation V of $\tilde{G}$ with diagonal action of the center, we obtain a projectively linear action of G on $\mathbb{P}(V)$. By assumption, the class is trivial upon restriction to abelian subgroups $A \subset G$. Thus, the A-actions are linear and have fixed points on $\mathbb{P}(V)$, i.e., $\mathbb{P}(V)$ satisfies **(A)**. On the other hand, the action is not (stably) linearizable, since

$$\mathrm{Am}^2(\mathbb{P}(V), G) \neq 0.$$

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Duncan (2016)

Let X be a del Pezzo surface of degree ≥ 3 with a generically free regular G-action. If X satisfies Condition **(A)** then X is G-unirational.

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  - $G = \mathrm{Q}_8$, via $\mathrm{Am}^3(G)$ discussed earlier.
- Which del Pezzo surfaces (of degree ≥ 3) with generically free regular G-actions are projectively G-unirational?

Unirationality

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- If X is a quadric with $X(k) \neq \emptyset$, project from a k-rational point, to obtain rationality.

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- If X is a quadric with $X(k) \neq \emptyset$, project from a k-rational point, to obtain rationality.
- If X is a (smooth) cubic surface of dimension with a $X(k) \neq \emptyset$, get two (sufficiently general) points, consider tangent hyperplane sections at these points, get a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$, by taking the third point of intersection.

Nonclosed fields and equivariant geometry: twisting

Consider

- G/k - algebraic group
- $T \to \mathrm{Spec}(k)$ a G-torsor
- X/k - (smooth) projective G-variety
- ${}^{T}X := (X \times T)/G$, with diagonal action, twist of X via T.

Nonclosed fields and equivariant geometry: twisting

Theorem (Duncan–Reichstein 2011)

Let G be a linear algebraic group and X a smooth projective G-variety, over k. Then the G-action on X is stably linearizable if and only if for every K/k every twist ${}^{T}X$ over K is stably rational.

Note: A similar statement for linearizability fails: there exist nonlinearizable but stably linearizable actions on quadrics of dimension ≥ 2.

Stable linearizability via subgroups

Duncan-Reichstein (2015)

Let $X \subset \mathbb{P}^n$, $n \geq 4$ be a smooth quadric, with a regular, generically free action of a finite group G. Then the G-action is stably linearizable if and only if the action of the 2-Sylow subgroup $Syl_2(G)$ of G is stably linearizable.

Standard constructions over BG

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First observations:

- G-unirationality is compatible with taking products;
- existence of a dominant G-equivariant rational map $X \dashrightarrow Y$ from a G-unirational X implies G-unirationality of Y.

Standard constructions over BG

Let G be a finite group and V a representation of G giving rise to a generically free G-action on a smooth quadric $X \subset \mathbb{P}(V)$. If there is a G-unirational irreducible $S \subset X$ then X is G-unirational.

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Here, we allow S to have a nontrivial generic stabilizer!

Indeed, consider

$$S \times \mathbb{P}(V) \dashrightarrow X,$$

sending $(s, p) \in S \times \mathbb{P}(V)$ to the second intersection point of X with the line through s and p.

Standard constructions over BG

Let $X \subset \mathbb{P}(V)$ be a G-invariant irreducible (possibly singular) cubic hypersurface of dimension ≥ 2 which is not a cone. Assume that the G-action on X is generically free and that $X^G \neq \emptyset$. Then X is G-unirational.

This is an application of Duncan–Reichstein.

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line through a general $p \in S$ intersects X in one point; the map is equivariant and dominant. Versions of the no-name lemma yield the claim.

Applications of Duncan-Reichstein

Stable linearizability from subgroups:

- There is a nonlinearizable $G = C_2^2 \times \mathfrak{S}_3$-action on a quadric threefold satisfying Condition (A). In particular, the $Syl_2(G)$-action is linearizable.

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  is birationally rigid. But $Syl_2(G)$ has fixed points.
- A $G = \mathfrak{D}_n$-action, with n odd, on a smooth quadric is stably linearizable. We do not know how to linearize some $\mathfrak{S}_3$-actions on 3-dimensional quadrics, e.g.,

$$x_1x_2 + x_3x_4 + x_5^2 = 0,$$

  with the faithful $\mathfrak{S}_3$-representation on x_1, x_2 and x_3, x_4.

Stable linearizability of quadric threefolds (Hassett-T. 2024)

Consider the action of $G \subset W(\mathsf{D}_5)$ on

$$X := \{\sum_{i=1}^{5} x_i^2 = 0\} \subset \mathbb{P}(V),$$

where V is the standard representation of $W(\mathsf{D}_5)$ via permutations and sign changes. Assume that

- Condition **(A)** holds, and
- G does not contain a subgroup $H \simeq \mathfrak{D}_4$, such that

  $$V|_H = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus V_2,$$

  where V_2 is the unique irreducible 2-dimensional representation of $\mathfrak{D}_4$, and χ_1, χ_2, χ_3 are pairwise distinct.

Then the G-action is stably linearizable.

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Let X be a smooth 3-dimensional quadric, with a generically free regular action of a finite group G satisfying Condition **(A)**. Then X is G-unirational.

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Let X be a smooth 3-dimensional quadric, with a generically free regular action of a finite group G satisfying Condition $\mathtt{(A)}$. Then X is G-unirational.

This settles all actions, not just those arising from subgroups of $W(\mathsf{D}_5)$.

Quadrics

Main steps:

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- Thus $X \subset \mathbb{P}(V)$, where
  (1) $V = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus V_2$,
  (2) $V = \chi \oplus V_2 \oplus V_2'$, or
  (3) $V = \chi \oplus V_4$,

  where V_2, V_2', V_4 are irreducible.

Quadrics

Combinatorial arguments, involving $\mathrm{B}^2(G)$, show that:

(1) Condition **(A)** for X implies Condition **(A)**, and G-unirationality, for the conic

$$C := \{x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{P}(\chi_1 \oplus \chi_2 \oplus \chi_3).$$

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(2) Condition **(A)** for X implies Condition **(A)** for one of the conics or the quadric surface in $\mathbb{P}(\chi \oplus V_2 \oplus V_2')$; again one has to address the issue of generic stabilizers.

(3) Condition **(A)** fails.

Smooth cubic threefolds

Automorphisms of smooth cubic threefolds have been classified (Wei-Yu 2020). Maximal noncyclic groups are:

	Cubic	Automorphism
X_1	$\sum_{i=j}^{5} x_j^3 = 0$	$C_3^4 \rtimes \mathfrak{S}_5$
X_2	$3(\sqrt{3}-1)x_1x_2x_3 + \sum_{i=1}^{5} x_j^3 = 0$	$((C_3^2 \rtimes C_3) \rtimes C_4) \times \mathfrak{S}_3$
X_5	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 = 0$	$\mathrm{PSL}_2(\mathbb{F}_{11})$
X_6	$\sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{5} x_i = 0$	$C_3 \times \mathfrak{S}_5$

Smooth cubic threefolds (Cheltsov-T.-Zhang 2025)

Our approach:

- List all (82) nonabelian groups acting on cubics, find all 5-dimensional representations giving rise to an action of a smooth cubic threefold, via **magma**;

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- G-invariant G-unirational hyperplane section, for 15 pairs.

Smooth cubic threefolds

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Assuming the Cassels–Swinnerton-Dyer conjecture the last three actions are unirational.

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There are many examples of such actions failing stable linearizability, via nontrivial

$$\mathrm{H}^1(G, \mathrm{Pic}(X)).$$

Intersections of two quadrics

Hassett-T. (2021)

Let $X_{2,2} \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics and $G \subseteq \mathrm{Aut}(X)$. The G-action on $X_{2,2}$ is linearizable if and only if $F_1(X)^G \neq \emptyset$.

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The following are equivalent:

- the G-action on $X_{2,2}$ is unirational,
- the G-action on $X_{2,2}$ satisfies condition **(A)**, i.e., every abelian subgroup of G fixes a point on $X_{2,2}$,
- $X^G \neq \emptyset$.

Summary

We discussed:

- invariants/obstructions in equivariant birational geometry
- G-unirationality of Fano threefolds